

On the critical points of the energy functional on vector fields of a Riemannian manifold

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Abstract

Given a compact Lie subgroup G of the isometry group of a compact Riemannian manifold M with a Riemannian connection ∇ , it is introduced a G -symmetrization process of a vector field of M and it is proved that the critical points of the energy functional

$$F(X) := \frac{\int_M \|\nabla X\|^2 dM}{\int_M \|X\|^2 dM}$$

on the space of G -invariant vector fields are critical points of F on the space of all vector fields of M , and that this inclusion may be strict in general. One proves that the infimum of F on \mathbb{S}^3 is not assumed by a \mathbb{S}^3 -invariant vector field. It is proved that the infimum of F on a sphere \mathbb{S}^n , $n \geq 2$, of radius $1/k$, is k^2 , and is assumed by a vector field invariant by the isotropy subgroup of the isometry group of \mathbb{S}^n at any given point of \mathbb{S}^n . It is proved that if G is a compact Lie subgroup of the isometry group of a compact rank 1 symmetric space M which leaves pointwise fixed a totally geodesic submanifold of dimension bigger than or equal to 1 then the infimum of F is assumed by a G -invariant vector field.

Finally, it is obtained a characterization of the spheres by proving that on a certain class of Riemannian compact manifolds M that contains rotationally symmetric manifolds and rank 1 symmetric spaces, with positive Ricci curvature Ric_M , F has the lower bound $\text{Ric}_M / (n - 1)$ among the G -invariant vector fields, where G is the isotropy subgroup of the isometry group of M at a point of M , and that this lower bound is attained if and only if M is a sphere of radius $1/\sqrt{\text{Ric}_M}$.

1 Introduction

Let M be a compact, orientable, n -dimensional, $n \geq 2$, C^∞ manifold with a Riemannian metric and let ∇ be the Riemannian connection of M .

In this paper we study the critical points of the energy of ∇ acting on the space $C^\infty(TM)$ of C^∞ vector fields of M with unit L^2 norm. Precisely, we study the critical points of the functional

$$F(X) = \int_M \|\nabla X\|^2 dM$$

on the space of vector fields $X \in C^\infty(TM)$ such that

$$\int_M \|X\|^2 dM = 1.$$

It is well known that the critical values of F are the eigenvalues of the so called rough Laplacian $-\operatorname{div} \nabla$ of M ([10]) and it follows from the spectral theory for linear elliptic operators that they constitute an increasing sequence $0 \leq \delta_1 < \delta_2 < \dots \rightarrow +\infty$ (counted with multiplicity) which are assumed by C^∞ eigenvector fields. Moreover, if M has no parallel vector fields then the infimum δ_1 of F is positive.

The search of geometric estimates of the spectrum of elliptic linear operators is an active topic of investigation in Geometric Analysis, the Laplacian being already a classical and well studied one ([9]). As to the study of the rough Laplacian operator, it seems to the authors that no attention has been paid so far. We have not found in the literature a description of its eigenvalues even in the simplest Riemannian spaces as the spheres.

In this paper we study the critical points of F on Riemannian compact manifolds admitting a nontrivial isometry group. A simple but fundamental idea here is to introduce a process of symmetrization of a vector field of M by a compact Lie subgroup G of the isometry group $\operatorname{Iso}(M)$ of M . We then use this symmetrization to prove that, apart some exceptional cases where the symmetrization process leads to zero vector fields, the critical points of F on the space of G -invariant vector fields are in the spectrum of F . What is somewhat surprising is that this inclusion may be strict, even for the infimum: We prove that the infimum of F in the unit sphere \mathbb{S}^3 is not realized by a \mathbb{S}^3 -invariant vector field, considering \mathbb{S}^3 as a Lie subgroup of $O(4)$; in other words, left invariant vector fields of \mathbb{S}^3 (with a bi-invariant metric) are critical points of F but are not ones with least energy. This raises the problem of whether the infimum of F is assumed by a G -invariant vector field for a given compact Lie subgroup G of the isometry group of M .

We prove that the orthogonal projection V of a nonzero vector v of \mathbb{R}^{n+1} on $T\mathbb{S}^n(1/k)$, where $\mathbb{S}^n(1/k)$ is a sphere of radius $1/k$, realizes the infimum

of F on $C^\infty(T\mathbb{S}^n(1/k))$ and that this infimum is k^2 . We note that V is invariant by the isotropy subgroup of $O(n+1)$ that leaves fixed the point $v/(k|v|) \in \mathbb{S}^n(1/k)$. We also prove that if M is a compact rank 1 symmetric space and if G is a Lie subgroup of $\text{Iso}(M)$ which leaves pointwise fixed a totally geodesic submanifold of dimension bigger than or equal to 1 then the infimum of F is realized by a G -invariant vector field.

In the last part of the paper we obtain a characterization of the sphere as a space where F attains the infimum of the energy on a certain class of Riemannian manifolds as explained next. We shall say that M is two point symmetric with center $p \in M$ if the isotropy subgroup $\text{Iso}_p(M)$ of $\text{Iso}(M)$ at p is isomorphic to the isotropy subgroup of the isometry group of a rank 1 symmetric space, that is, $\text{Iso}_p(M)$ is isomorphic to some of the following Lie groups: $O(n)$, $U(1) \times U(n-1)$, $Sp(1) \times Sp(n-1)$ or $\text{Spin}(9)$ ([3] and Definition 11 ahead). Note that when $\text{Iso}_p(M) = O(n)$ then M is a rotationally symmetric space (see [2]). Symmetric spaces of rank 1 satisfy the so called two point homogeneous property and hence are also known as two point homogeneous spaces.

We prove that if M is a two point symmetric space with center p and with Ricci curvature Ric_M satisfying $\text{Ric}_M \geq (n-1)k^2$, then the infimum of F on the space of $\text{Iso}_p(M)$ -invariant vector fields is bigger than or equal to k^2 and the equality holds if and only if M is a sphere of radius $1/k$ (Theorem 12).

2 A general result

Let M be a compact Riemannian manifold of dimension $n \geq 2$. We choose on the full isometry group $\text{Iso}(M)$ of M a fixed left invariant Riemannian metric. We consider on any Lie subgroup of $\text{Iso}(M)$ the left invariant Riemannian metric induced by the one of $\text{Iso}(M)$.

Let G be a compact Lie subgroup of $\text{Iso}(M)$. Given a vector field $V \in C^\infty(TM)$, the G -symmetrization of V is the vector field V_G defined by setting, at a given $p \in M$,

$$\langle V_G(p), u \rangle = \frac{1}{\text{Vol}(G)} \int_G \langle (dg_p)^{-1} V(g(p)), u \rangle dg$$

where $u \in T_p M$. Note that $V_G \in C^\infty(TM)$ is G -invariant, that is, $V_G(g(p)) = dg_p(V_G(p))$ for all $p \in M$ and $g \in G$. Moreover, by the linearity of $\text{div} \nabla$ and of the integration process we have

$$(\text{div} \nabla V)_G = \text{div} \nabla V_G.$$

In particular, if V satisfies $\operatorname{div} \nabla V = -\lambda V$ then $\operatorname{div} \nabla V_G = -\lambda V_G$. We call V_G the G -mean of V .

We observe that depending on M and G it may happen that $V_G \equiv 0$. This happens with any vector field V on a rank 1 compact symmetric space M if G is the full isometry group of isometries of M (this is consequence of Lemma 7). We prove:

Proposition 1 *Let M be a compact n -dimensional Riemannian manifold and G a compact Lie subgroup of $\operatorname{Iso}(M)$. Then the eigenvalues and eigenvectors of F restrict to the subspace of G -invariant vector fields of $C^\infty(TM)$ are also eigenvalues and eigenvectors of F on $C^\infty(TM)$.*

We need the following result:

Lemma 2 *On the hypothesis of the Proposition 1 assume that $W \in C^\infty(TM)$ satisfies $W_G \equiv 0$. Let V be a G -invariant vector field. Then*

$$\int_M \langle W(x), V(x) \rangle dx = 0.$$

Proof. Since the elements of G are isometries and V is G -invariant we have, for all $g \in G$

$$\begin{aligned} \int_M \langle W(x), V(x) \rangle dx &= \int_M \langle W(g(x)), V(g(x)) \rangle dx \\ &= \int_M \langle dg_x^{-1}(W(g(x))), dg_x^{-1}(V(g(x))) \rangle dx \\ &= \int_M \langle dg_x^{-1}(W(g(x))), V(x) \rangle dx. \end{aligned}$$

It follows from Fubini's theorem that

$$\begin{aligned} \int_M \langle W(x), V(x) \rangle dx &= \frac{1}{\operatorname{Vol}(G)} \int_G \int_M \langle dg_x^{-1}(W(g(x))), V(x) \rangle dx dg \\ &= \int_M \left\langle \frac{1}{\operatorname{Vol}(G)} \int_G dg_x^{-1}(W(g(x))) dg, V(x) \right\rangle dx \\ &= \int_M \langle W_G(x), V(x) \rangle dx = 0. \end{aligned}$$

proving the lemma. ■

Proof of the Proposition 1. Let X an eigenvector of $\operatorname{div} \nabla$ on the space of the G -invariant vector fields associated to the eigenvalue λ . Then

$$\int_M \langle -\operatorname{div} \nabla X, W \rangle dx = \lambda \int_M \langle X, W \rangle dx,$$

for all G -invariant vector field W . For proving that

$$\int_M \langle -\operatorname{div} \nabla X, V \rangle dx = \lambda \int_M \langle X, V \rangle dx,$$

holds for any given vector field $V \in C^\infty(TM)$ we write $V = Z + V_G$ with $Z = V - V_G$ so that

$$\int_M \langle -\operatorname{div} \nabla X, V \rangle dx = \int_M \langle -\operatorname{div} \nabla X, Z \rangle dx + \int_M \langle -\operatorname{div} \nabla X, V_G \rangle dx.$$

Noting that Z has zero G -mean and and that $\operatorname{div} \nabla X$ is a G -invariant because X is we have, by Lemma 2, that the first term of the hand side of the equality above is zero and then

$$\begin{aligned} \int_M \langle -\operatorname{div} \nabla X, V \rangle dx &= \int_M \langle -\operatorname{div} \nabla X, V_G \rangle dx = \lambda \int_M \langle X, V_G \rangle dx \\ &= \lambda \int_M \langle X, V - Z \rangle dx = \lambda \int_M \langle X, V \rangle dx \end{aligned}$$

proving the proposition. ■

Remark 3 *Considering a bi-invariant metric on the unit sphere \mathbb{S}^3 with the Lie group structure, \mathbb{S}^3 is a Lie subgroup of the isometry group $O(4)$ of \mathbb{S}^3 . It follows from Proposition 1 that the \mathbb{S}^3 -invariant vector fields are eigenvectors of the rough Laplacian of \mathbb{S}^3 . Clearly the \mathbb{S}^3 -invariant vector fields are the left (and right) invariant vector fields of \mathbb{S}^3 . The orbits of a left invariant vector field constitute a Hopf fibration of \mathbb{S}^3 .*

We have that the energy of a left invariant vector field is 2. Indeed, the Bochner-Yano formula for a vector field $X \in C^\infty(T\mathbb{S}^3)$, namely,

$$\begin{aligned} \int_{\mathbb{S}^3} |\nabla X|^2 dx &= \int_{\mathbb{S}^3} \left(\operatorname{Ric}(X, X) + 2 |\operatorname{Kill}(X)|^2 - (\operatorname{div} X)^2 \right) dx \\ &= \int_{\mathbb{S}^3} \left(2 |X|^2 + 2 |\operatorname{Kill}(X)|^2 - (\operatorname{div} X)^2 \right) dx, \end{aligned}$$

where $\operatorname{Kill}(X)$ is the $(0, 2)$ -tensor

$$\operatorname{Kill}(X)(U, V) = \frac{\langle \nabla_U X, V \rangle + \langle \nabla_V X, U \rangle}{2}, \quad U, V \in C^\infty(T\mathbb{S}^3)$$

(see [11]), when X is a Killing field, since $\operatorname{Kill}(X) \equiv 0$ and $\operatorname{div} X \equiv 0$, gives

$$F(X) = \frac{\int_{\mathbb{S}^3} |\nabla X|^2 dx}{\int_{\mathbb{S}^3} |X|^2 dx} = 2.$$

As we will see in the next section, the infimum of F on \mathbb{S}^n is 1, for all n . Thus, the \mathbb{S}^3 -invariant vector fields do not realize the infimum of energy on \mathbb{S}^3 .

3 The infimum of the energy on a sphere

In this section we determine the infimum of F in the case that M is a sphere. Denoting by $\mathbb{S}^n(1/k)$ the sphere of radius $1/k$ in \mathbb{R}^{n+1} we prove:

Theorem 4 *The infimum of F on $C^\infty(T\mathbb{S}^n(1/k))$, $n \geq 2$, is k^2 , and is assumed by the orthogonal projection on $T\mathbb{S}^n(1/k)$ of a constant nonzero vector field of \mathbb{R}^{n+1} .*

We need some preliminary lemmas.

Lemma 5 *Let X be a vector field of $\mathbb{S}^n(1/k)$ with zero G -mean ($X_G \equiv 0$), where G is the isotropy subgroup of $\text{Iso}(\mathbb{S}^n(1/k))$ that leaves fixed a point $v \in \mathbb{S}^n(1/k)$. Then the function $f(p) := \langle X(p), v \rangle$, $p \in \mathbb{S}^n(1/k)$, has zero mean in $\mathbb{S}^n(1/k)$ that is,*

$$\int_{\mathbb{S}^n(1/k)} f(x) dx = 0.$$

Proof. Using the formula of coarea to integrate f on $\mathbb{S}^n(1/k)$ along the level sets of $h : \mathbb{S}^n(1/k) \rightarrow \mathbb{R}$, $h(p) = d(p, v)$, where d is the distance in $\mathbb{S}^n(1/k)$, since $\|\text{grad } h\| = 1$, we have

$$\int_{\mathbb{S}^n(1/k)} f(x) = \int_0^\pi \left(\int_{h^{-1}(t)} f(x) \right) dt. \quad (1)$$

We note that $h^{-1}(t)$ is a geodesic sphere \mathbb{S}_t^{n-1} of $\mathbb{S}^n(1/k)$ centered at v and with radius t and that these spheres are the orbits of G . Given $t \in (0, \pi/k)$, choose $p \in h^{-1}(t)$, and denote by H the subgroup of isotropy of G at p . Let

$$\psi : \frac{G}{H} \rightarrow \mathbb{S}_t^{n-1}$$

be given by $\psi(gH) = g(p)$. We note that, up to a factor multiplying the metric of G , ψ is an isometry. Setting

$$\tilde{f} := f \circ \psi : \frac{G}{H} \rightarrow \mathbb{R}$$

we then obtain

$$\int_{h^{-1}(t)} f(x) dx = \int_{\frac{G}{H}} \tilde{f}(gH) d(gH). \quad (2)$$

Let

$$\phi : G \rightarrow \frac{G}{H},$$

be the projection of G over G/H , and set $\bar{f} := \tilde{f} \circ \phi$. Using the coarea formula to integrate \bar{f} on G along the fibers of ϕ , we obtain

$$\begin{aligned} \int_{g \in G} \bar{f}(g) dg &= \int_{\frac{G}{H}} \left(\int_{gH} \frac{1}{\|\text{Jac } \phi\|} \bar{f}(gh) d(gh) \right) d(gH) \\ &= \int_{\frac{G}{H}} \left(\int_{gH} \bar{f}(gh) d(gh) \right) d(gH). \end{aligned}$$

Since \bar{f} is constant on gH , $g \in G$, we get

$$\int_G \bar{f}(g) dg = \text{Vol}(H) \int_{\frac{G}{H}} \tilde{f}(gH) d(gH)$$

or by (2)

$$\int_{h^{-1}(t)} f(x) dx = \frac{1}{\text{Vol}(H)} \int_G \bar{f}(g) dg. \quad (3)$$

But

$$\begin{aligned} \int_G \bar{f}(g) dg &= \int_G \tilde{f}(gH) dg \\ &= \int_G (f \circ \psi)(gH) dg = \int_G f(g(p)) dg \\ &= \int_G \langle X(g(p)), v \rangle dg = \int_G \langle dg_p^{-1} X(g(p)), dg_p^{-1} v \rangle dg \\ &= \int_G \langle dg_p^{-1} X(g(p)), v \rangle dg = \langle X_G(p), v \rangle = 0. \end{aligned}$$

Therefore from (3)

$$\int_{h^{-1}(t)} f = 0,$$

for any $t \in (0, \pi/k)$, and this, with (1), proves the lemma. ■

Lemma 6 *Let $V \in C^\infty(T\mathbb{S}^n(1/k))$, $n \geq 2$, with*

$$\int_{\mathbb{S}^n(1/k)} |V|^2 = 1$$

be given. Let G be the isotropy subgroup of $\text{Iso}(\mathbb{S}^n(1/k))$ at some point of $\mathbb{S}^n(1/k)$. If V is nonzero and has zero G -mean, then $F(V) \geq (n-1)k^2$.

Proof. Denote by ∇ the Riemannian connection of $\mathbb{S}^n(1/k)$. Write

$$V = \sum_{l=1}^{n+1} a_l e_l,$$

where $\{e_l\}$ is an orthonormal basis of the \mathbb{R}^{n+1} . Fix $p \in \mathbb{S}^n(1/k)$ and let $\{E_j\}$ be an orthonormal frame of $\mathbb{S}^n(1/k)$ on a neighborhood of p . Then, for each i we have

$$\nabla_{E_i} V = \sum_{l=1}^{n+1} E_i(a_l) e_l - \langle V, E_i \rangle k p$$

and thus

$$\begin{aligned} \langle \nabla V, \nabla V \rangle &= \sum_{i=1}^n \langle \nabla_{E_i} V, \nabla_{E_i} V \rangle \\ &= \sum_{i=1}^n \left(-\langle V, E_i \rangle^2 k^2 + \sum_{l=1}^{n+1} \langle E_i(a_l) e_l, E_i(a_l) e_l \rangle \right) \\ &= -k^2 |V|^2 + \sum_{l=1}^{n+1} |\text{grad}(a_l)|^2. \end{aligned}$$

We then have

$$F(V) = -k^2 + \sum_{l=1}^{n+1} \int_{\mathbb{S}^n(1/k)} |\text{grad}(a_l)|^2.$$

By Lemma 5 the functions a_l have zero mean on $\mathbb{S}^n(1/k)$. Since the first positive eigenvalue of $\mathbb{S}^n(1/k)$ is equal to nk^2 , from Sobolev inequality we obtain

$$\begin{aligned} F(V) &\geq -k^2 + nk^2 \sum_{l=1}^{n+1} \int_{\mathbb{S}^n(1/k)} |a_l|^2 \\ &= -k^2 + nk^2 \sum_{l=1}^{n+1} \int_{\mathbb{S}^n(1/k)} |V|^2 = (n-1) k^2. \end{aligned}$$

■

Lemma 7 Assume that a Lie subgroup H of $O(n) = \text{Iso}(\mathbb{S}^n(1/k))$ acts transitively on $\mathbb{S}^n(1/k)$. Then

$$\int_H \langle hu, v \rangle dh = 0 \tag{4}$$

for any fixed vectors $u, v \in \mathbb{S}^n(1/k)$.

Proof. Let $u, v \in \mathbb{S}^n(1/k)$ be given. There is $g \in H$ such that $g(v) = -v$. Since the left translation $L_{g^{-1}} : H \rightarrow H$, $L_{g^{-1}}(h) = g^{-1}h$ is an orientation preserving isometry of H we have

$$\int_H \langle hu, v \rangle dh = \int_H \langle g^{-1}hu, v \rangle (L_{g^{-1}})^* dh = \int_H \langle hu, gv \rangle dh = - \int_H \langle hu, v \rangle dh$$

which proves (4). ■

The next lemma will also be used to prove Theorem 12. We make use of the following terminology: An orbit of highest dimension of a compact Lie group G acting on a compact manifold M is called a principal orbit (except to the exceptional orbits, see [4]). We say that G acts with cohomogeneity one if the principal orbits of G have codimension 1. The principal orbits of G foliates a open dense subset of M whose complementary has zero $\dim M$ -dimensional measure ([4]).

Lemma 8 *Let M^n be a compact, orientable Riemannian manifold. Let G be a compact Lie subgroup of $\text{Iso}(M)$ acting with cohomogeneity one on M . Assume moreover that the subgroup of isotropy of G at any point of a principal orbit of G acts transitively (by the derivative) on the spheres centered at origin of the tangent space of the orbit at the point. Then any G -invariant vector field is orthogonal to the principal orbits of G .*

Proof. Let $p \in M$ be such that $G(p)$ is a principal orbit of G and let $v \in T_p G(p)$ be any fixed vector. Since X is G -invariant we have

$$\langle X(p), v \rangle = \langle dg_p^{-1}(X(g(p))), v \rangle = \langle X(g(p)), dg_p v \rangle$$

for all $g \in G$, so that

$$\langle X(p), v \rangle = \frac{1}{\text{Vol}(G)} \int_G \langle X(g(p)), dg_p v \rangle dg.$$

Denoting by H be the isotropy subgroup of G at p we have by coarea formula

$$\int_G \langle X(g(p)), dg_p v \rangle dg = \int_{G/H} \left(\int_{gH} \langle X((gh)(p)), d(gh)_p v \rangle d(gh) \right) d(gH).$$

Moreover

$$\begin{aligned} \int_{gH} \langle X((gh)(p)), d(gh)_p v \rangle d(gh) &= \int_H \langle X(g(p)), d(gh)_p v \rangle dh \\ &= \int_H \langle dg_p^{-1} X(g(p)), dh_p v \rangle dh = 0 \end{aligned}$$

by the previous lemma, since the action $h \mapsto dh_p$ of H on $T_p G(p)$ is transitive on the spheres of $T_p G(p)$. Then

$$\langle X(p), v \rangle = \frac{1}{\text{Vol}(G)} \int_G \langle X(g(p)), dg_p v \rangle dg = 0.$$

This implies that $X(p) \in (T_p G(p))^\perp$ since v is arbitrary, concluding with the proof of the lemma. ■

Proof of Theorem 4. A calculation shows that if $X \in C^\infty(T\mathbb{S}^n(1/k))$ is the orthogonal projection on $T\mathbb{S}^n(1/k)$ of a vector $v \in \mathbb{R}^{n+1}$ then $-\text{div } \nabla X = k^2 X$. We will prove that k^2 is the infimum of F and hence proving the theorem. We may assume that $v \in \mathbb{S}^n(1/k)$.

Let G be the isotropy subgroup of $\text{Iso}(\mathbb{S}^n(1/k))$ at v and $W \in C^\infty(T\mathbb{S}^n(1/k))$ assuming the infimum of F . It follows from Lemma 6 that if W has zero G -mean, then $F(W) \geq (n-1)k^2$ and the theorem is proved in this case. We may then assume that W_G is non zero. Since $\text{div } W_G = (\text{div } W)_G$, F assumes its infimum at W_G too.

By the Lemma 8, W_G is orthogonal to the orbits of G , which are geodesic spheres centered at v . We then have $W_G = \langle W_G, \text{grad } s \rangle \text{grad } s$, where s is the distance in $\mathbb{S}^n(1/k)$ to v . Define $h \in C^2([0, \pi/k])$ by $h(t) = \langle W_G, \text{grad } s \rangle(x)$, where $x \in \mathbb{S}^n(1/k)$ is such that $t = s(x)$. Since W_G is G -invariant h is well defined and we have $W_G = h(s) \text{grad } s$.

If $\phi \in C^2([0, \pi/k])$ is a primitive of h and $f : \mathbb{S}^n(1/k) \rightarrow \mathbb{R}$ is defined by $f(x) = \phi(s(x))$ then we have

$$\text{grad } f = \phi' \text{grad } s = h \text{grad } s = W_G.$$

Applying Reilly's formula to f (see [8]) we obtain

$$\begin{aligned} \int_{\mathbb{S}^n(1/k)} (\Delta f)^2 dx &= \int_{\mathbb{S}^n(1/k)} \text{Ric}_{\mathbb{S}^n(1/k)}(\text{grad } f, \text{grad } f) dx \\ &\quad + \int_{\mathbb{S}^n(1/k)} |\text{Hess}(f)|^2 dx \\ &= \int_{\mathbb{S}^n(1/k)} \text{Ric}_{\mathbb{S}^n(1/k)}(W_G, W_G) dx + \int_{\mathbb{S}^n(1/k)} |\text{Hess}(f)|^2 dx, \end{aligned}$$

where Δf and $|\text{Hess}(f)|$ are the Laplacian and the norm of the Hessian of f . Note that $|\text{Hess}(f)| = |\nabla W_G|$ and then, since $(\Delta f)^2 \leq n |\text{Hess}(f)|^2$,

assuming that $|W_G|_{L^2} = 1$, we obtain

$$\begin{aligned} & (n-1) \int_{\mathbb{S}^n(1/k)} |\nabla W_G|^2 dx \\ & \geq \int_{\mathbb{S}^n(1/k)} |W_G|^2 \operatorname{Ric}_{\mathbb{S}^n(1/k)} \left(\frac{W_G}{|W_G|}, \frac{W_G}{|W_G|} \right) dx \geq (n-1)k^2. \end{aligned}$$

It follows that $F(W_G) \geq k^2$, proving the theorem. ■

4 The infimum of the energy on a rank 1 compact symmetric space

In this section we prove:

Theorem 9 *Let M be a rank 1 compact symmetric space. Let G be a compact Lie subgroup of $\operatorname{Iso}(M)$ that leaves pointwise fixed a totally geodesic submanifold of M with dimension bigger than or equal to 2. Then the infimum of the energy in M is assumed by a G -invariant vector field.*

If V is a vector field on M and if $h : M \rightarrow M$ is a diffeomorphism, we denote by V^h the h -related vector field to V , that is, $V^h(p) = (dh_p)^{-1}(V(h(p)))$, $p \in M$. We use the following lemma.

Lemma 10 *Let $p \in M$, $v \in T_p M$ and $V \in C^\infty(TM)$ be given. Assume that $g(p) = p$ and $dg_p(v) = v$ for all $g \in G$. Given $q \in M$ assume that $h \in \operatorname{Iso}(M)$ is such that $h(p) = q$ and $dh_q^{-1}(V(q)) = v$. Let $x_n \in M$ be a sequence converging to p . Then*

$$V^h(p) = \lim_{n \rightarrow \infty} \frac{\int_G dg_{x_n}^{-1}(V^h(g(x_n)))}{\operatorname{Vol}(G(x_n))},$$

where $\operatorname{Vol}(G(x_n))$ is the k -dimensional Hausdorff measure of $G(x_n)$, $k = \dim G(x_n) \geq 0$.

Proof. We will prove that

$$\left\langle V^h(p), Z(p) \right\rangle = \left\langle \lim_{n \rightarrow \infty} \frac{\int_G dg_{x_n}^{-1}(V^h(g(x_n)))}{\operatorname{Vol}(G(x_n))}, Z(p) \right\rangle, \quad Z \in C^\infty(TM).$$

Given $Z \in C^\infty(TM)$ we have, for a given n ,

$$\left\langle \frac{\int_G dg_{x_n}^{-1}(V^h(g(x_n)))}{\operatorname{Vol}(G(x_n))}, Z(x_n) \right\rangle = \frac{\int_G \langle dg_{x_n}^{-1}(V^h(g(x_n))), Z(x_n) \rangle}{\operatorname{Vol}(G(x_n))}$$

so that

$$\begin{aligned} \inf_{g \in G} \left\langle dg_{x_n}^{-1}(V^h(g(x_n))), Z(x_n) \right\rangle &\leq \frac{\int_G \langle dg_{x_n}^{-1}(V^h(g(x_n))), Z(x_n) \rangle \omega}{\text{Vol}(G(x_n))} \\ &\leq \sup_{g \in G} \left\langle dg_{x_n}^{-1}(V^h(g(x_n))), Z(x_n) \right\rangle. \end{aligned}$$

Letting $n \rightarrow \infty$ we obtain

$$\begin{aligned} \inf_{g \in G} \left\langle dg_p^{-1}(V^h(g(p))), Z(p) \right\rangle &\leq \left\langle \lim_{n \rightarrow \infty} \frac{\int_G dg_{x_n}^{-1}(V^h(g(x_n))) dg}{\text{Vol}(G(x_n))}, Z(p) \right\rangle \\ &\leq \sup_{g \in G} \left\langle dg_p^{-1}(V^h(g(p))), Z(p) \right\rangle, \end{aligned}$$

and, since $g(p) = p$,

$$\begin{aligned} \inf_{g \in G} \left\langle dg_p^{-1}(V^h(p)), Z(p) \right\rangle &\leq \left\langle \lim_{n \rightarrow \infty} \frac{\int_G dg_{x_n}^{-1}(V^h(g(x_n))) dg}{\text{Vol}(G(x_n))}, Z(p) \right\rangle \\ &\leq \sup_{g \in G} \left\langle dg_p^{-1}(V^h(p)), Z(p) \right\rangle. \end{aligned}$$

Since

$$dg_p^{-1}(V^h(p)) = dg_p^{-1}(v) = v = V^h(p)$$

for all $g \in G$ it follows that

$$\left\langle V^h(p), Z(p) \right\rangle \leq \left\langle \lim_{n \rightarrow \infty} \frac{\int_G dg_{x_n}^{-1}(V^h(g(x_n))) dg}{\text{Vol}(G(x_n))}, Z(p) \right\rangle \leq \left\langle V^h(p), Z(p) \right\rangle$$

proving the lemma. ■

Proof of Theorem 9. Let $V \in C^\infty(TM)$ be a critical point of F with unit L^2 -norm. Since V is non zero, there is $q \in M$ such that $V(q) \neq 0$. Let G be a Lie subgroup of $\text{Iso}(M)$ that leaves pointwise fixed a totally geodesic submanifold T of M , $\dim T \geq 1$. Choose a $p \in T$ and a nonzero vector $v \in T_p T$. We have $g(p) = p$ for all $g \in G$ and we claim that $dg_p(v) = v$ for all $g \in G$ too. Up to a multiplication of v by a positive real number, we may assume that v is contained on a normal geodesic ball of $T_p T$. Set $w = dg_p(v)$. Then, since $\exp_p v \in T$, where \exp_p is the Riemannian exponential, we have

$$\exp_p v = g(\exp_p v) = \exp_{g(p)} dg_p(v) = \exp_p w$$

and then $v = w$.

Since M is a compact rank 1 symmetric space, there is an isometry h of the M such that $h(p) = q$ and $dh_p^{-1}(V(q)) = v$. Let V^h be the vector field h -related to V . We claim that V_G^h is not identically zero. Indeed, taking a sequence $x_n \in M$ converging to p , we have, by the Lemma 10

$$V^h(p) = \lim_{n \rightarrow \infty} \frac{\int_G dg_{x_n}^{-1}(V^h(g(x_n)))}{\text{Vol}(G(x_n))} = \lim_{n \rightarrow \infty} \frac{V_G^h(x_n)}{\text{Vol}(G(x_n))}.$$

Hence, if $V_G^h \equiv 0$ then $V^h(p) = 0$, a contradiction! This proves the theorem. \blacksquare

5 A characterization of spheres

Rotationally symmetric manifolds are well known and much used as models on comparison theorems on Geometric Analysis. We consider here a generalization of such manifolds which we call *two point symmetric manifolds*:

Definition 11 *We say that a Riemannian n -dimensional manifold M , $n \geq 2$, is two point symmetric with center $p \in M$ if the isotropy subgroup $G := \text{Iso}_p(M)$ of the isometry group of M at p is isomorphic to the isotropy subgroup of the isometry group of a two point homogeneous space S at any point in S .*

We note that the above definition is a natural generalization of the way used in [2] to define rotationally symmetric spaces. We also observe that this definition is equivalent to the requirement that given points $p_1, p_2, p_3, p_4 \in M$ that belong to a common geodesic sphere of M centered at p , if $d(p_1, p_2) = d(p_3, p_4)$, where d is the Riemannian distance in M , then there is an isometry $i \in G$ such that $i(p_1) = p_3$ and $i(p_2) = p_4$ (see [1], [3]). It is also known that this two point homogenous characterization of M around p is equivalent to the isotropy of G at any point $x \in M$ of a principal orbit $G(x)$ of G acting transitively on the Euclidean spheres centered at the origin of $T_x G(x)$ ([3]).

We also recall that a two point homogeneous space is isometric to a rank 1 symmetric space. Accordingly to the symmetric space classification, it follows that a Riemannian manifold M is two point homogenous with center $p \in M$ if $\text{Iso}_p(M)$ is isomorphic to one of the following Lie groups: $O(n)$, $U(1) \times U(n)$, $Sp(1) \times Sp(n)$ or $\text{Spin}(9)$ (see [3]). When $\text{Iso}_p(M) = O(n)$ the space M is rotationally symmetric.

Theorem 12 *Let M^n be a compact two point symmetric space with center p , $n \geq 2$, with positive Ricci curvature Ric_M and assume that $\text{Ric}_M \geq (n-1)k^2$, $k > 0$. Set $G = \text{Iso}_p(M)$. Then the infimum of F on the G -invariant vector fields is bigger than or equal to k^2 and the equality holds if and only if M is a sphere of radius $1/k$.*

Proof. Denote by $s : M \rightarrow \mathbb{R}$ the distance in M to p . Note that the level sets of s are geodesic spheres centered at p and that the mean curvature and the norm of the second fundamental form of these geodesic spheres depend only on s . Set $l = \max s$.

Let $V \in C^\infty(TM)$ a G -invariant vector field such that $F(V)$ is the positive infimum of F on the space of G -invariant vector fields. Since the subgroup of isotropy of G at a point p of a principal orbit of G acts transitively (by the derivative) on the spheres centered at origin of $T_p G(p)$ it follows from Lemma 8 that V may be written on the form $V = \langle V, \text{grad } s \rangle \text{grad } s$. Define h, f and ϕ as in Theorem 4. The same reasoning used in this theorem allows us to conclude that $F(V) \geq k^2$, proving the first part of the theorem. By Theorem 4 we know that if M is a sphere of radius $1/k$ then $F(V) = k^2$. We now prove the converse. Thus, assume that $F(V) = k^2$. From the inequality obtained from Reilly's formula in Theorem 4 we obtain

$$(\Delta f)^2 = n |\text{Hess}(f)|^2. \quad (5)$$

Given $s \in (0, l)$ denote by $|B|(s)$ and $H(s)$ the norm of the second fundamental form and the mean curvature of the geodesic sphere at a distance s of p , with respect to the unit normal vector pointing to the center p . Since $f = \phi \circ s$, straightforward calculations give

$$(\Delta f)^2 = (\phi'')^2 - 2(n-1)\phi''\phi'H + (n-1)^2(\phi')^2 H^2$$

and

$$|\text{Hess}(f)|^2 = (\phi'')^2 + (\phi')^2 |B|^2.$$

Then (5) is equivalent to

$$(\phi'' + H\phi')^2 = (\phi')^2 \left[-\frac{n|B|^2}{n-1} + nH^2 \right] \quad (6)$$

which implies that

$$-\frac{|B|^2}{n-1} + H^2 \geq 0 \quad (7)$$

since ϕ' cannot be identically zero (otherwise V would be, which is not the case). But, denoting by λ_i the principal curvatures of the geodesic spheres, we have

$$\begin{aligned} -\frac{|B|^2}{n-1} + H^2 &= \frac{-\left(\sum_{i=1}^{n-1} \lambda_i^2\right)}{n-1} + \frac{\left(\sum_{j=1}^{n-1} \lambda_j\right)^2}{(n-1)^2} \\ &= -\sum_{i,j=1, i < j}^n (\lambda_i - \lambda_j)^2 \leq 0 \end{aligned}$$

so that

$$-\frac{|B|^2}{n-1} + H^2 = 0,$$

and, from (6),

$$\phi'' + H\phi' = 0. \quad (8)$$

Since

$$V = \text{grad } f = (h \circ s) \text{grad } s$$

and V is an infimum of F we have

$$-\text{div } \nabla (h(s) \text{grad } s) = k^2 h(s) \text{grad } s.$$

A calculation gives

$$\text{div } \nabla (h \text{grad } s) = \left(h''(s) - (n-1) H(s) h'(s) - |B|^2(s) h(s) \right) \text{grad } s$$

so that

$$h'' - (n-1) H h' - |B|^2 h = -k^2 h. \quad (9)$$

But from (8) we have

$$h'' - (n-1) H h' - |B|^2 h = h'' + (n-1) \left[H^2 - |B|^2 / (n-1) \right] h = h''$$

and hence

$$h''(s) = -k^2 h(s).$$

It follows that h is of the form

$$h(s) = A \cos(sk) + B \sin(ks)$$

and, since $\phi' = h$,

$$\phi(s) = \frac{A}{k} \cos(sk) + \frac{B}{k} \sin(ks).$$

From $f(x) = \phi(s(x))$ we then obtain using (8)

$$\Delta f = \phi'' + \phi' \Delta s = \phi'' - (n-1)H\phi' = \phi'' + (n-1)\phi'' = n\phi'' = -nk^2\phi = -nk^2f.$$

Hence nk^2 is a eigenvalue of the usual Laplacian. Under the hypothesis $\text{Ric}_M \geq (n-1)k^2$ this implies that M is a sphere of radius $1/k$ ([7]), finishing the proof of the theorem. ■

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